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The application of uniform-slender-body theory to the motion of two ships in shallow water

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The techniques of uniform-slender-body theory are employed to investigate the hydrodynamic forces and moments acting on a moving ship in shallow water and the interaction forces between two such ships on parallel courses. Of particular interest is the verification by these methods of the validity of the solutions by matched asymptotic expansions constructed by previous authors. The free surface is assumed rigid and each ship is modelled as a slender body of revolution located midway between two closely spaced parallel planes. The velocity potential due to the presence of a single ship is represented as the potential due to singularities distributed along a portion of the axis inside the body, together with appropriate image singularities outside the body. The boundary condition on the body leads to a linear integral equation for the density of singularities, which is solved using the asymptotic analysis discussed by Geer (1975). The sinkage force and trimming moment on the vessel are computed. When two ships are moving on parallel courses, appropriate interaction potentials are introduced in a manner similar to that for a single ship and the integral equations resulting from the application of the boundary condition are solved asymptotically. The interaction forces and moments between the ships are computed and compared with some experimental and other theoretical results.

1. Introduction

This paper employs the techniques of uniform-slender-body theory to investigate the hydrodynamic forces acting on a moving ship in shallow water, in particular the interaction forces between two such ships on parallel courses. The determination of these forces is of importance in many practical situations, such as the manoeuvring of ships in shallow, congested harbours and the passing of two ships in canals.

The general problem of the motion of ships in restricted waters has received the attention recently of several investigators, e.g. Fujino (1976), King (1977), Newman (1969), Tuck (1978), Tuck & Newman (1974), and Yeung (1978). Most of these investigators use (in one form or another) an aerodynamic equivalence principle (see Tuck 1978 and Newman & Wu 1973), which essentially models the flow in the far field as the flow past a two-dimensional airfoil. In the near field, a two-dimensional problem in a plane containing a cross-section of the ship is obtained by neglecting changes along the length of the ship. After solving these two boundary-value problems, the ideas of inner and outer asymptotic expansions are used to match the solutions to these two problems and thus provide an approximate solution valid over most of the flow field, i.e. except near the ends of the ship. Here the small parameter involved in the asymptotic expansions is the shallowness parameter $\epsilon = h/L$, where h is the depth of the water and L is the length of the ship.

The method of solution used here is more direct than that discussed above, in the sense that it involves a single, relatively simple representation of the solution, which will be uniformly valid over the entire flow field, including the ends of the ships. Thus the need to compute several different expansions of the solution is avoided, while at the same time the significance of 'end effects' in the determination of the forces on the ships is taken into account. A significant feature of the rigorous asymptotic analysis presented here is that Yeung's solution by matched asymptotic expansions is shown to be embedded, perhaps better than might be expected analytically, in this more precise solution of the posed boundary-value problem. This property of verification is hardly diminished by the need to restrict consideration to axisymmetric bodies, a common feature of the type of slender-body analysis employed in this paper.

In § 2, the problem for the motion of a single ship in shallow water is formulated and the representation of the solution is constructed. For this case, the velocity potential due to the presence of the ship is represented as the potential due to singularities distributed along a portion of an axis inside the ship, together with an appropriate distribution of image singularities outside the fluid region. The boundary condition on the surface of the body leads to a linear integral equation to determine the densities and location of the singularities inside the ship. This equation is solved in § 3, using the asymptotic analysis discussed by Geer (1975) and Handelsman & Keller (1967). In particular, explicit expressions are derived for the leading terms in the expansions of the density functions, which involve only the body geometry and purely numerical constants, which can be evaluated recursively. The results of §§ 2 and 3 are then used in § 4 to compute the pressure coefficient, making due allowance for end effects, and hence the sinkage force and trimming moment on a single vessel due to its motion in shallow water.

In § 5, the problem of two ships moving on parallel courses in shallow water is formulated and solved by an analysis similar to that in § 3. Here the representation of the full potential involves not only potentials for each ship alone but also appropriate interaction potentials whose density functions are related to the cross-flow velocity induced by the second ship through factors which depend only on the geometry of the first vessel and vice versa and which can again be evaluated recursively. Thus the solution presented here is not only more exact than that given by Yeung (1978), with whose work comparison is made, but also has this most attractive and practically useful feature of being calculated recursively rather than requiring the solution of an integral equation or, equivalently, an infinite set of simultaneous equations. After using the results of § 5 to compute the sway force and yaw moment exerted by the fluid on each vessel, there follows a brief description of suitably chosen examples and discussion of their relevance to the physical problem being modelled.

Although it is assumed below that the vessels have rounded ends, the subsequent retention of only the leading terms means that pointed bows or fin-like sterns can be simulated in the form of cone- or spindle-shaped ends. The application of the methods used here to non-circular cross-sections requires the latter to be considered as perturbations of semicircular ones. Nevertheless it is expected that such a future investigation will produce useful results.

2. Problem formulation for a single vessel

Consider a slender vessel of length L moving at speed U in the fore–aft direction in an inviscid fluid of depth h. Let $OXYZ$ be Cartesian axes fixed in space with $Z = 0$ the free surface, Z measured vertically downwards and OX the line through the bow and stern of the vessel. Introduce dimensionless co-ordinates x, y, z fixed in the vessel so that the bow and stern are at $(1,0,0)$ and $(0,0,0)$ respectively, i.e. (see figure 1)

$$Lx = X-Ut, \quad Ly = Y, \quad Lz = Z. \tag{2.1}$$

Thus the time t is measured from the instant when the stern coincides with the fixed origin. The shallowness of the water and the slenderness of the body are assumed to be small and of the same order of magnitude. Then $\epsilon = h/L$ is a suitable small parameter and, restricting consideration to bodies of revolution, the surface $\mathscr{B}$ of the vessel is given by

$$r = (y^2+z^2)^{\frac{1}{2}} = \epsilon(S(x))^{\frac{1}{2}} \quad (0 \leqslant x \leqslant 1), \tag{2.2}$$

where $S(0) = 0 = S(1)$ and $\max S(x) < 1$. It is assumed that $S(x)$ is analytic on $0 \leqslant x \leqslant 1$ and can be expanded about the end points as follows:

$$S(x) = \sum_{n=1}^{\infty} c_n x^n, \quad S(x) = \sum_{n=1}^{\infty} d_n(1-x)^n \tag{2.3}$$

with $c_1 \neq 0 \neq d_1$, i.e. non-zero radii of curvature at the bow and stern. This geometrical description is that given by Geer (1975) and is a necessary preliminary to the construction of an asymptotic solution for the velocity potential ϕ, exploiting the slenderness parameter ϵ as described by Hendelsman & Keller (1967).

Following Yeung (1978), it is assumed throughout that the free surface is rigid, which implies that wave effects are neglected. This is known to be plausible if the depth Froude number $U/(gh)^{\frac{1}{2}}$ is $o(\epsilon)$, where g is the acceleration due to gravity. Thus the infinite-gravity limit of the more general problem, where wave effects are important, appears to be the leading-order problem corresponding to a low-speed perturbation analysis. This rigid-free-surface condition reduces the problem to the determination of the symmetrical flow past the body and its image above the free surface, enclosed between parallel walls a distance $2h$ apart (see figure 1).

In terms of the co-ordinates x, y, z fixed in the body, the absolute velocity $\nabla\phi(x, y, z)$ of the fluid particles due to the motion of the vessel is determined by the equations

$$\nabla^2\phi = 0 \quad \text{throughout the fluid;} \tag{2.4}$$

$$\frac{\partial \phi}{\partial z} = 0 \quad \text{at} \quad z = \pm\epsilon; \tag{2.5}$$

$$\frac{\partial}{\partial n}(\phi - ULx) = 0 \quad \text{at} \quad r = \epsilon(S(x))^{\frac{1}{2}} \quad (0 < x < 1); \tag{2.6}$$

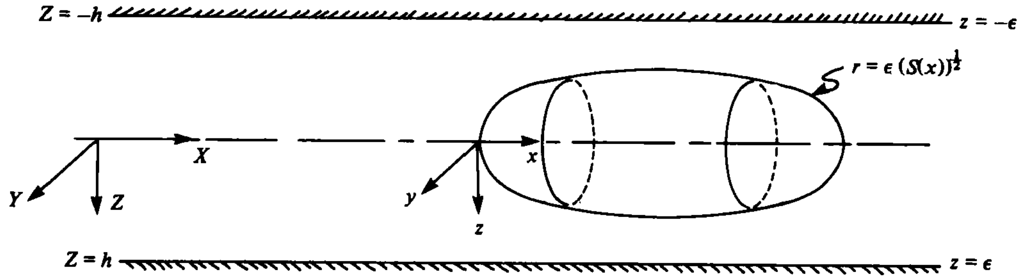

FIGURE 1. The hull of a slender ship in shallow water and its reflection above the free surface modelled as a slender body of revolution between two closely spaced parallel planes. The $OXYZ$ co-ordinate system is fixed in space, while the non-dimensional $oxyz$ co-ordinate system is fixed in the body.

where $\partial/\partial n$ denotes differentiation normal to the surface of the body. It is a straightforward matter to satisfy all conditions except (2.6) by representing ϕ as the superposition of potentials due to unknown distributions of singularities on the axis of the body.

The Green's function $G_n(x,y,z)$ that has singularity

$$\mathscr{R}\frac{(y+iz)^n}{(x^2+y^2+z^2)^{n+\frac{1}{2}}}$$

at the origin and satisfies (2.4), (2.5), is given by

$$G_n(x,y,z) = \mathscr{R}\sum_{m=-\infty}^{\infty}\frac{(y+iz+2mi\epsilon)^n}{[x^2+y^2+(z+2m\epsilon)^2]^{n+\frac{1}{2}}} \quad (n \geqslant 1), \tag{2.7}$$

$$G_0(x,y,z) = (x^2+y^2+z^2)^{-\frac{1}{2}}+\sum_{m=1}^{\infty}\left\{[x^2+y^2+(z-2m\epsilon)^2]^{-\frac{1}{2}}+[x^2+y^2+(z+2m\epsilon)^2]^{-\frac{1}{2}}-\frac{1}{m\epsilon}\right\}.$$

Thus an infinite array of similar image singularities is required to construct each G_n, with the constant term inserted into the $n=0$ series to ensure convergence of the m-summation. In the limit $\epsilon \to 0$, each array of singularities becomes, in some sense, a line singularity.

For the corresponding problem in unbounded fluid, Handelsman & Keller (1967) constructed a velocity potential of the form

$$UL\epsilon^2\int_{\alpha(\epsilon)}^{\beta(\epsilon)}\frac{g_0(\xi,\epsilon)\,d\xi}{[(x-\xi)^2+y^2+z^2]^{\frac{1}{2}}},$$

where

$$g_0(x,\epsilon) = \tfrac{1}{4}S'(x)+O(\epsilon^2\ln\epsilon), \tag{2.8}$$

$$\int_{\alpha(\epsilon)}^{\beta(\epsilon)} g_0(x,\epsilon)\,dx = 0, \tag{2.9}$$

$$\left.\begin{aligned}\frac{\alpha(\epsilon)}{c_1} &= (\tfrac{1}{2}\epsilon)^2-c_2(\tfrac{1}{2}\epsilon)^4+(c_1c_3+2c_2^2)(\tfrac{1}{2}\epsilon)^6+O(\epsilon^8),\\ \frac{1-\beta(\epsilon)}{d_1} &= (\tfrac{1}{2}\epsilon)^2-d_2(\tfrac{1}{2}\epsilon)^4+(d_1d_3+2d_2^2)(\tfrac{1}{2}\epsilon)^6+O(\epsilon^8).\end{aligned}\right\} \tag{2.10}$$

Here α and β, which measure the extent of the singularity distribution within the body, are determined by the requirement that $[(x-\xi)^2+\epsilon^2S(x)]^{\frac{1}{2}}$ have an expansion in powers of ϵ which remains regular when $x \to 0$ with $\xi = \alpha(\epsilon)$ and when $x \to 1$ with $\xi = \beta(\epsilon)$.

The existence of the boundaries at $z = \pm\epsilon$ creates reflected velocities which, near the body, are of order U. These are evidently asymmetric and can be cancelled at the surface $\mathscr{B}$ by suitable distributions of higher-order singularities

$$\left\{\frac{r^n \cos n\theta}{[(x-\xi)^2+r^2]^{n+\frac{1}{2}}};\, n \geqslant 1\right\} \quad (y = r\cos\theta, \quad z = r\sin\theta)$$

as described by Geer (1975), who showed that the nth density function must be of the form $[\xi-\alpha(\epsilon)]^n\,[\beta(\epsilon)-\xi]^n\, g_n(\xi,\epsilon)$ and the end points of integration are given, for all n, by (2.10). The multiply reflected velocities are completely taken into account by using the Green's functions defined by (2.7), whence the velocity potential can be written in the form

$$\phi = UL \sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{1}{2})\,\epsilon^{n+2}}{\Gamma(\frac{1}{2})\,n!} \int_{\alpha(\epsilon)}^{\beta(\epsilon)} [\xi-\alpha(\epsilon)]^n\,[\beta(\epsilon)-\xi]^n\, g_n(\xi,\epsilon)\, G_n(x-\xi,y,z)\, d\xi. \tag{2.11}$$

Here the gamma-function factors are inserted for convenience, ϵ^2 corresponds to the cross-sectional area $(\epsilon L)^2\pi S(x)$ of the vessel and ϵ^n anticipates the order of magnitude of the nth density function which has an expansion involving positive powers of ϵ^2 and $\ln\epsilon$, of which only the leading term is sought in the present calculation, i.e.

$$g_n(x,\epsilon) \sim g_n(x) + O(\epsilon^2 \log\epsilon) \quad (n \geqslant 0). \tag{2.12}$$

Flux considerations determine that (2.9) remains true. The image singularities in $G_n(X-\xi,y,z)$ play no role in determining $\alpha(\epsilon), \beta(\epsilon)$ because, for $m \neq 0$, the distance of the singularity from a point on $\mathscr{B}$, namely $[(x-\xi)^2+\epsilon^2(S+4mS^{\frac{1}{2}}\sin\theta+4m^2)]^{\frac{1}{2}}$, cannot become zero, its minimum value being $\epsilon[2m-(S(x))^{\frac{1}{2}}]$. In the next section, it will be shown that the leading terms $\{g_n(x);\, n \geqslant 0\}$ of the density functions can be determined in a recursive manner.

3. Solution for the single vessel

Condition (2.6) requires that

$$2(S(x))^{\frac{1}{2}}\left[\cos\theta\frac{\partial\phi}{\partial y}+\sin\theta\frac{\partial\phi}{\partial z}\right] - \epsilon S'(x)\left(\frac{\partial\phi}{\partial x}-UL\right) = 0$$

at

$$y+iz = r\,e^{i\theta}, \quad r = \epsilon(S(x))^{\frac{1}{2}} \quad (0 < x < 1),$$

which, on substitution of (2.11), yields the following equation for the determination of $\{g_n(x,\epsilon);\, n \geqslant 0\}$:

$$\sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{1}{2})\,\epsilon^{n+2}}{\Gamma(\frac{1}{2})\,n!} \int_{\alpha(\epsilon)}^{\beta(\epsilon)} [\xi-\alpha(\epsilon)]^n\,[\beta(\epsilon)-\xi]^n\, g_n(\xi,\epsilon)\, \mathscr{R} \sum_{m=-\infty}^{\infty} \left\{\frac{-2n(S(x))^{\frac{1}{2}}\,e^{i\theta}\,\overline{Q_m}^{\,n-1}\,\epsilon^{n-1}}{[(x-\xi)^2+\epsilon^2\,|Q_m|^2]^{n+\frac{1}{2}}} + \frac{(2n+1)\,\overline{Q_m}^{\,n}\,\epsilon^{n+1}[(S(x))^{\frac{1}{2}}\,(e^{i\theta}Q_m+e^{-i\theta}\overline{Q_m}) - S'(x)\,(x-\xi)]}{[(x-\xi)^2+\epsilon^2\,|Q_m|^2]^{n+\frac{3}{2}}}\,d\xi\right\} - \epsilon S'(x) = 0 \quad (0 < x < 1), \tag{3.1}$$

where the complex-valued functions $\{Q_m\}$ are defined by

$$Q_m(x,\theta) = (S(x))^{\frac{1}{2}}\,e^{-i\theta} - 2mi \quad (-\infty < m < \infty) \tag{3.2}$$

and an overbar denotes the complex conjugate. On the left-hand side of (3.1), consider first the $m = 0$ terms. The contribution for $n = 0$ is

$$\epsilon^3 \int_{\alpha(\epsilon)}^{\beta(\epsilon)} g_0(\xi,\epsilon) \frac{[2S(x) - S'(x)(x-\xi)]}{[(x-\xi)^2 + \epsilon^2 S(x)]^{\frac{3}{2}}} d\xi = 2\epsilon \frac{d}{dx} \int_{\alpha(\epsilon)}^{\beta(\epsilon)} \frac{(x-\xi) g_0(\xi,\epsilon)\, d\xi}{[(x-\xi)^2 + \epsilon^2 S(x)]^{\frac{1}{2}}}$$

$$= 2\epsilon \frac{d}{dx}\left[\int_{\alpha(\epsilon)}^{x} g_0(\xi)\, d\xi - \int_{x}^{\beta(\epsilon)} g_0(\xi)\, d\xi\right] + O(\epsilon^3 \ln \epsilon) = 4\epsilon g_0(x) + O(\epsilon^3 \ln \epsilon),$$

as shown by Handelsman & Keller (1967). The $n \geqslant 1$ terms yield

$$\sum_{n=1}^{\infty} \frac{\Gamma(n+\frac{1}{2})\, \epsilon^{2n+1}}{\Gamma(\frac{1}{2})\, n!} [S(x)]^{\frac{1}{2}n} \cos n\theta \int_{\alpha(\epsilon)}^{\beta(\epsilon)} [\xi - \alpha(\epsilon)]^n [\beta(\epsilon) - \xi]^n g_n(\xi,\epsilon)$$

$$\times \frac{[-2n(x-\xi)^2 + 2(n+1)\epsilon^2 S(x) - (2n+1)\epsilon^2 S'(x)(x-\xi)]}{[(x-\xi)^2 + \epsilon^2 S(x)]^{n+\frac{3}{2}}} d\xi$$

$$= 2\epsilon \sum_{n=1}^{\infty} g_n(x) \left[\frac{x(1-x)}{(S(x))^{\frac{1}{2}}}\right]^n \cos n\theta + O(\epsilon^3 \ln \epsilon)$$

according to Geer (1975), after correcting two misprints. In his equation (5.4), the power of 2 should be $(2n-1)$ and the n factor should be deleted. Thus, for each n, the leading term arises from integration in a small neighbourhood of the point $\xi = x$, as is evident when the analytic density function is expanded in a Taylor series.

For the $m \neq 0$ terms, the dominant contributions can be calculated more easily because $|Q_m| \geqslant 2m - (S(x))^{\frac{1}{2}} > 2m - 1$. Indeed, if $x(1-x) \neq O(\epsilon)$, then the range of values of $(x-\xi)/\epsilon$ is asymptotically $-\infty$ to ∞ and hence

$$\sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{1}{2})\, \epsilon^{2n+1}}{\Gamma(\frac{1}{2})\, n!} \int_{\alpha(\epsilon)}^{\beta(\epsilon)} [\xi - \alpha(\epsilon)]^n [\beta(\epsilon) - \xi]^n g_n(\xi,\epsilon)$$

$$\times \left\{\frac{-2n(S(x))^{\frac{1}{2}} e^{i\theta} \overline{Q_m}^{\,n-1}}{[(x-\xi)^2 + \epsilon^2 |Q_m|^2]^{n+\frac{1}{2}}} + \frac{(2n+1)\overline{Q_m}^{\,n} \epsilon^2 [(S(x))^{\frac{1}{2}} (e^{i\theta} Q_m + e^{-i\theta}\overline{Q_m}) - S'(x)(x-\xi)]}{[(x-\xi)^2 + \epsilon^2 |Q_m|^2]^{n+\frac{3}{2}}}\right\} d\xi$$

$$= \sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{1}{2})}{\Gamma(\frac{1}{2})\, n!} \epsilon x^n (1-x)^n g_n(x) \int_{-\infty}^{\infty} \left\{\frac{(2n+1)\overline{Q_m}^{\,n+1} (S(x))^{\frac{1}{2}} e^{-i\theta}}{[\xi^2 + |Q_m|^2]^{n+\frac{3}{2}}}\right.$$

$$\left. + \frac{d}{d\xi}\left[\frac{(S(x))^{\frac{1}{2}} e^{i\theta} \overline{Q_m}^{\,n-1} \xi - \epsilon S'(x) \overline{Q_m}^{\,n}}{[\xi^2 + |Q_m|^2]^{n+\frac{1}{2}}}\right]\right\} d\xi + O(\epsilon^3 \ln \epsilon)$$

$$= 2\epsilon (S(x))^{\frac{1}{2}} e^{-i\theta} \sum_{n=0}^{\infty} \frac{x^n (1-x)^n}{Q_m^{\,n+1}} g_n(x) + 2\epsilon (S(x))^{\frac{1}{2}} e^{i\theta} \frac{g_0(x)}{Q_m} + O(\epsilon^3 \ln \epsilon).$$

If $x(1-x) = O(\epsilon)$, then the range of values of $(x-\xi)/\epsilon$ is asymptotically semi-infinite but the above result can be substituted into (3.1) with negligible error. Regarding the functions $\{g_n(x); n \geqslant 0\}$ as of order unity, the error at the nth term is evidently of order $\epsilon^{n+\frac{3}{2}}$ since $S(x) = O(\epsilon)$. This estimate can, by more detailed analysis, be improved for the $n = 0$ terms to order ϵ^2. On the left-hand side of the above equation, the $n = 0$ terms are asymptotically equal to

$$2\epsilon g_0(x)\, \mathscr{R} \int_0^1 \frac{\epsilon^2 (S(x))^{\frac{1}{2}} e^{-i\theta}\, \overline{Q_m}}{[(x-\xi)^2 + \epsilon^2 |Q_m|^2]^{\frac{3}{2}}} d\xi = 2\epsilon g_0(x)\, \mathscr{R}\left[\frac{(S(x))^{\frac{1}{2}} e^{-i\theta} (\xi - x)}{Q_m [(x-\xi)^2 + \epsilon^2 |Q_m|^2]^{\frac{1}{2}}}\right]_0^1.$$

Then, adding to this the similar expression obtained for the corresponding negative value of m, as is required in (3.1), the result is

$$2\epsilon g_0(x)\,\mathscr{R}\left[\frac{(S(x))^{\frac{1}{2}}e^{-i\theta}(\xi-x)}{Q_m[(x-\xi)^2+\epsilon^2|Q_m|^2]^{\frac{1}{2}}}+\frac{(S(x))^{\frac{1}{2}}e^{-i\theta}(\xi-x)}{Q_{-m}[(x-\xi)^2+\epsilon^2|Q_{-m}|^2]^{\frac{1}{2}}}\right]_0^1.$$

On substituting (3.2), the expression in curly brackets is readily seen to be uniformly of order $S(x) = O(\epsilon)$ as ξ varies from 0 to 1. Similarly, on the right-hand side, the $n = 0$ terms yield

$$4\epsilon g_0(x)\,\mathscr{R}\left\{(S(x))^{\frac{1}{2}}e^{-i\theta}\left(\frac{1}{Q_m}+\frac{1}{Q_{-m}}\right)\right\} = 8\epsilon g_0(x)\,\mathscr{R}\left[\frac{S(x)\,e^{-2i\theta}}{S(x)\,e^{-2i\theta}+4m^2}\right].$$

Thus the algebra has demonstrated the physically obvious fact that the reflections of the velocity field are negligible at the ends of the body.

On collecting results, the leading terms of (3.1) yield

$$\tfrac{1}{2}S'(x) = 2g_0(x)+\sum_{n=0}^{\infty}g_n(x)\left[\frac{x(1-x)}{(S(x))^{\frac{1}{2}}}\right]^n\cos n\theta+2g_0(x)\,\mathscr{R}\sum_{m=1}^{\infty}\left(\frac{1}{Q_m}+\frac{1}{Q_{-m}}\right)(S(x))^{\frac{1}{2}}e^{-i\theta}$$
$$+\sum_{n=1}^{\infty}x^n(1-x)^n g_n(x)\,\mathscr{R}\sum_{m=1}^{\infty}\left(\frac{1}{Q_m^{\,n+1}}+\frac{1}{Q_{-m}^{\,n+1}}\right)(S(x))^{\frac{1}{2}}e^{-i\theta}. \tag{3.3}$$

It is now desirable to express the summations over m as Fourier series in θ and this is achieved using the relation

$$\sum_{m=1}^{\infty}\frac{2W}{W^2+4m^2} = \sum_{s=1}^{\infty}(-1)^{s-1}\left(\frac{W}{2}\right)^{2s-1}\zeta(2s) \tag{3.4}$$

(Gradsteyn & Ryzhik 1965) and its derivatives with respect to W. Here $\zeta(2s)$ denotes the Riemann zeta function, i.e.

$$\zeta(2s) = \sum_{m=1}^{\infty}\frac{1}{m^{2s}}.$$

Equation (3.3) can now be rewritten as

$$\tfrac{1}{2}S'(x) = 2g_0(x)+g_0(x)\sum_{s=1}^{\infty}(-\tfrac{1}{4})^{s-1}[S(x)]^s\zeta(2s)\cos 2s\theta+\sum_{n=1}^{\infty}g_n(x)\left(\frac{x(1-x)}{(S(x))^{\frac{1}{2}}}\right)^n$$
$$\times\left\{\cos n\theta+\sum_{s=[\frac{1}{2}n]+1}^{\infty}\frac{(-1)^{n+s-1}}{2^{2s-1}}(S(x))^s\frac{(2s-1)!\,\zeta(2s)}{n!\,(2s-n-1)!}\cos(2s-n)\theta\right\}, \tag{3.5}$$

where the square brackets here denote the integral part. Equating the terms independent of θ, it is found that

$$g_0(x) = \tfrac{1}{4}S'(x), \tag{3.6}$$

which is the same result as that obtained in the absence of the boundaries at $z = \pm\epsilon$. The s-summation, with factor $g_0(x)$, in (3.5) corresponds to multiple reflections of the infinite-fluid solution. The additional terms are due to the higher-order singularities and their multiple reflections. On equating Fourier coefficients in (3.5), the appearance of only even cosines in the above-mentioned s-summation means that

$$g_{2p-1}(x) = 0 \quad (p \geqslant 1), \tag{3.7}$$

whilst the functions $\{g_{2p}(x); p \geqslant 1\}$ are determined, after substituting (3.6), by the infinite set of equations:

$$g_{2p}(x)\left[\frac{x(1-x)}{S(x)}\right]^{2p} - 2\sum_{q=1}^{\infty} g_{2q}(x)\,[x(1-x)]^{2q}\,(-\tfrac{1}{4})^{p+q}\frac{(2q+2p-1)!}{(2q)!\,(2p-1)!}\zeta(2p+2q)$$
$$= (-\tfrac{1}{4})^{p}\,\zeta(2p)\,S'(x) \quad (p \geqslant 1). \quad (3.8)$$

Note that the q-summation becomes negligible at the ends of the body. The solution for $g_{2p}(x)$ $(p \geqslant 1)$ must be of the form

$$g_{2p}(x) = (-1)^{p}\left[\frac{S(x)}{x(1-x)}\right]^{2p} S'(x)\sum_{r=0}^{\infty} a_{pr}[S(x)]^{2r} \quad (p \geqslant 1), \quad (3.9)$$

in which the coefficients $\{a_{pr}\}$ are independent of x and can be determined by substituting (3.9) into (3.8) and equating coefficients of powers of $S(x)$. Thus

$$\sum_{r=0}^{\infty} a_{pr}[S(x)]^{2r} - \frac{2}{4^p}\sum_{q=1}^{\infty}\sum_{r=0}^{\infty} a_{qr}[S(x)]^{2q+2r}\frac{(2q+2p-1)!\,\zeta(2p+2q)}{4^q(2q)!\,(2p-1)!} = \frac{1}{4^p}\zeta(2p) \quad (p \geqslant 1)$$

and hence

$$\left.\begin{aligned} a_{p0} &= 4^{-p}\,\zeta(2p) \quad (p \geqslant 1);\\ a_{ps} &= 4^{-p}2\sum_{q=1}^{s}\frac{a_{q,s-q}}{4^q}\frac{(2q+2p-1)!}{(2q)!\,(2p-1)!}\zeta(2p+2q) \quad (p \geqslant 1,\ s \geqslant 1).\end{aligned}\right\} \quad (3.10)$$

Not surprisingly, it is seen that the first term on the right-hand side of (3.9) is, for each p, due to the pth reflection of the infinite fluid solution. The coefficients $\{a_{ps}\}$ can be calculated recursively; numerical computation showed that $0 < a_{ps} < 1$ for all $p \geqslant 1$, $s \geqslant 0$ and $a_{ps} \to 0$ as $s \to \infty$ for each $p \geqslant 1$. Hence each series in (3.9) is convergent for all $S(x)$ such that $0 < S(x) < 1$, as required.

Thus, the velocity potential ϕ for a single vessel in shallow water is given by equations (2.11), (2.7) and (2.12), where α and β are given by (2.10) and the functions $\{g_n\}$ are determined by equations (3.6), (3.7), (3.9) and (3.10).

4. Pressure on the single vessel

It is now convenient to consider the time-independent situation in which velocities are expressed relative to the axes $O'xyz$ fixed in the body. Then, by Bernoulli's theorem, the pressure P in the fluid is given by

$$P = P_\infty + \tfrac{1}{2}\rho\left[U^2 - \left\{\operatorname{grad}\left(\frac{\phi}{L} - Ux\right)\right\}^2\right], \quad (4.1)$$

where P_∞ denotes the pressure when $x^2+y^2 \to \infty$ and ρ is the fluid density. In particular, the pressure P_s at the bow and stern stagnation points is given by

$$P_s = P_\infty + \tfrac{1}{2}\rho U^2. \quad (4.2)$$

Seeking to evaluate the pressure at the surface $\mathscr{B}$ $(r = \epsilon(S(x))^{\frac{1}{2}};\ 0 \leqslant x \leqslant 1)$, it is first noted that, according to (2.6), the normal derivative of $(\phi - ULx)$ is zero. The tangential components of velocity are

$$\frac{2(S(x))^{\frac{1}{2}}\left(\frac{\partial\phi}{\partial x} - UL\right) + \epsilon S'(x)\left(\cos\theta\frac{\partial\phi}{\partial y} + \sin\theta\frac{\partial\phi}{\partial z}\right)}{L\{\epsilon^2[S'(x)]^2 + 4S(x)\}^{\frac{1}{2}}}$$

and, in an x = constant plane,

$$\left(-\sin\theta\frac{\partial\phi}{\partial y} + \cos\theta\frac{\partial\phi}{\partial z}\right)\Big/L,$$

where the derivatives of ϕ are evaluated on $\mathscr{B}$. On substituting (2.11), the first component above can be written as

$$\frac{2U(S(x))^{\frac{1}{2}}}{\{\epsilon^2[S'(x)]^2 + 4S(x)\}^{\frac{1}{2}}}\left\{-1 + \frac{d}{dx}\sum_{n=0}^{\infty}\frac{\Gamma(n+\frac{1}{2})\,\epsilon^{2n+2}}{\Gamma(\frac{1}{2})\,n!}\int_{\alpha(\epsilon)}^{\beta(\epsilon)}[\xi - \alpha(\epsilon)]^n[\beta(\epsilon) - \xi]^n g_n(\xi,\epsilon)\right.$$
$$\left.\times\mathscr{R}\sum_{m=-\infty}^{\infty}\frac{Q_m{}^n}{[(x-\xi)^2 + \epsilon^2|Q_m|^2]^{n+\frac{1}{2}}}d\xi\right\},$$

which evidently is identically zero at $x = 0, 1$ as required in (4.2). The expression in curly brackets is

$$-1 + \epsilon^2\frac{d}{dx}\int_{\alpha(\epsilon)}^{\beta(\epsilon)} g_0(\xi,\epsilon)\sum_{m=-\infty}^{\infty}\frac{1}{[(x-\xi)^2 + \epsilon^2|Q_m|^2]^{\frac{1}{2}}}d\xi + \frac{d}{dx}\sum_{n=1}^{\infty}\frac{\Gamma(n+\frac{1}{2})\,\epsilon^{2n+2}}{\Gamma(\frac{1}{2})\,n!}$$
$$\times\int_{\alpha(\epsilon)}^{\beta(\epsilon)}[\xi - \alpha(\epsilon)]^n[\beta(\epsilon) - \xi]^n g_n(\xi,\epsilon)\,\mathscr{R}\sum_{m=-\infty}^{\infty}\frac{\overline{Q_m}^n}{[(x-\xi)^2 + \epsilon^2|Q_m|^2]^{n+\frac{1}{2}}}d\xi$$
$$= -1 + \tfrac{1}{2}\epsilon\frac{d}{dx}\int_0^1 g_0(\xi)\int_{-\infty}^{\infty}\frac{d\nu}{[(x-\xi)^2 + \nu^2]^{\frac{1}{2}}}d\xi + O(\epsilon^2)$$
$$+\epsilon^2\frac{d}{dx}\sum_{p=1}^{\infty}\frac{1}{2p}[x(1-x)]^{2p}g_{2p}(x)\,\mathscr{R}\sum_{m=-\infty}^{\infty}\frac{1}{Q_m{}^{2p}}$$
$$= -1 - \epsilon\int_0^1\int_0^{\infty}\frac{g_0(\xi)\,d\nu(x-\xi)\,d\xi}{[(x-\xi)^2 + \nu^2]^{\frac{3}{2}}} + O(\epsilon^2)$$
$$= -1 - \epsilon\,⨍_0^1\frac{g_0(\xi)\,d\xi}{x-\xi} + O(\epsilon^2).$$

Here, the methods of the previous section have been used to estimate the ξ-integrals for $n \geqslant 1$, the remaining m-summations being convergent. However, for the $n = 0$ terms, the m-summation must be estimated, leaving a ξ-integral. What happens effectively is that, in contrast to the previous sections, the leading terms in the above x-derivatives have an $(x - \xi)$ factor which reduces the dominance of the contributions from the neighbourhood of $\xi = x$, making a different calculation necessary for the $n = 0$ terms for which the whole range of values of ξ is important.

The second tangential component of velocity is, on substituting (2.11), equal to

$$U\sum_{n=0}^{\infty}\frac{\Gamma(n+\frac{1}{2})\,\epsilon^{n+2}}{\Gamma(\frac{1}{2})\,n!}\int_{\alpha(\epsilon)}^{\beta(\epsilon)}[\xi - \alpha(\epsilon)]^n[\beta(\epsilon) - \xi]^n g_n(\xi,\epsilon)$$
$$\times\mathscr{R}\sum_{m=-\infty}^{\infty}\left\{\frac{n(\epsilon\overline{Q_m})^{n-1}\,i\,e^{i\theta}}{[(x-\xi)^2 + \epsilon^2|Q_m|^2]^{n+\frac{1}{2}}} - \frac{(2n+1)\,(\epsilon\overline{Q_m})^n\,2m\epsilon\cos\theta}{[(x-\xi)^2 + \epsilon^2|Q_m|^2]^{n+\frac{3}{2}}}\right\}d\xi.$$

When $S(x) \to 0$, i.e. $x \to 0$ or 1, the m-summation becomes, using (3.2),

$$\mathscr{R} \sum_{m=-\infty}^{\infty} \left\{ \frac{n(2mi\epsilon)^{n-1} i\, e^{i\theta}}{[(x-\xi)^2+4m^2\epsilon^2]^{n+\frac{1}{2}}} - \frac{(2n+1)\,(2mi\epsilon)^n\, 2m\epsilon\cos\theta}{[(x-\xi)^2+4m^2\epsilon^2]^{n+\frac{3}{2}}} \right\}.$$

This is identically zero if n is even, and for $n = 2p-1$ reduces to

$$-\frac{\sin\theta}{|x-\xi|^3}\delta_{1p} + 2\sin\theta(-1)^p \sum_{m=1}^{\infty} \frac{(2p-1)\,(2m\epsilon)^{2p-2}}{[(x-\xi)^2+4m^2\epsilon^2]^{2p-\frac{1}{2}}} \quad (p \geqslant 1),$$

where δ_{1p} is a Kronecker delta. Thus, if this expression is to be uniformly zero as required, the density functions $g_{2p-1}(x,\epsilon)$ must be identically zero for all $p \geqslant 1$. This is consistent with the earlier result (3.7) that the leading terms $\{g_{2p-1}(x);\, p \geqslant 1\}$ vanish and could reasonably have been anticipated from the boundary conditions.

For $S(x) \neq 0$ on $\mathscr{B}$, the second tangential component of velocity can be estimated by the methods used for the first component, which yield

$$-\tfrac{1}{2}U\epsilon\int_0^1 g_0(\xi)\int_{-\infty}^{\infty} \frac{\nu\cos\theta\, d\nu\, d\xi}{[(x-\xi)^2+\nu^2]^{\frac{3}{2}}} + O(U\epsilon^2) + \frac{U}{(S(x))^{\frac{1}{2}}}\frac{d}{d\theta}\sum_{p=1}^{\infty}\frac{\Gamma(2p+1)\,\epsilon^{4p+1}}{\Gamma(\frac{1}{2})\,(2p)!}$$
$$\times \int_{\alpha(\epsilon)}^{\beta(\epsilon)} [\xi-\alpha(\epsilon)]^{2p}\,[\beta(\epsilon)-\xi]^{2p}\, g_{2p}(\xi,\epsilon)\,\mathscr{R}\sum_{m=-\infty}^{\infty}\frac{\overline{Q_m}^{2p}}{[(x-\xi)^2+\epsilon^2\,|Q_m|^2]^{2p+\frac{1}{2}}}\,d\xi$$
$$= U\epsilon\sum_{p=1}^{\infty}[x(1-x)]^{2p}\, g_{2p}(x)\,\mathscr{R}\sum_{m=-\infty}^{\infty}\frac{ie^{-i\theta}}{Q_m^{2p+1}} + O(U\epsilon^2).$$

On substituting these results into (4.1), it is seen that the pressure at the surface $\mathscr{B}$ is given by

$$(P)_{\mathscr{B}} = P_\infty + \tfrac{1}{2}\rho U^2\left\{1 - \frac{4S(x)}{\epsilon^2[S'(x)]^2+4S(x)}\left[1+2\epsilon\,⨍_0^1\frac{g_0(\xi)\,d\xi}{x-\xi}\right] + O(\epsilon^2)\right\}$$
$$= P_\infty + \frac{\rho U^2}{\epsilon^2[S'(x)]^2+4S(x)}\left\{\tfrac{1}{2}\epsilon^2[S'(x)]^2 - 4\epsilon\, S(x)\,⨍_0^1\frac{g_0(\xi)\,d\xi}{x-\xi} + O(\epsilon^2)\right\}. \quad (4.3)$$

Here, the $S'(x)$ terms are only significant when $S(x)$ is small. Hence the sinkage force F_s, that is, the downward force exerted by the fluid on the 'wetted' half of the surface $\mathscr{B}$ due to the motion of the vessel, is given by

$$F_S = -L^2\int_0^\pi d\theta\int_0^1 dx\,\epsilon(S(x))^{\frac{1}{2}}\sin\theta[(P)_{\mathscr{B}}-P_\infty]$$
$$= \tfrac{1}{2}\rho U^2h^2\int_0^1\frac{(S(x))^{\frac{1}{2}}\,dx}{S(x)+\frac{1}{4}\epsilon^2[S'(x)]^2}\left[S(x)\,⨍_0^1\frac{S'(\xi)\,d\xi}{x-\xi} - \tfrac{1}{8}\epsilon[S'(x)]^2 + O(\epsilon)\right\}$$
$$= \tfrac{1}{2}\rho U^2h^2\left\{\int_0^1 (S(x))^{\frac{1}{2}}\,⨍_0^1\frac{S'(\xi)\,d\xi}{x-\xi}\,dx + O(\epsilon)\right\} \quad (4.4)$$

after substituting (4.3) and (3.6). Apart from scaling factors, the leading term of (4.4) and the first term of equation (48) given by Yeung (1978) are respectively the same as expressions (6.12) and (6.10) of Tuck (1966) with F_h set equal to zero in his earlier equation (5.8). In addition, the trimming moment about the line $x = \frac{1}{2}, z = 0$ is given by

$$M_T = -\tfrac{1}{2}\rho U^2h^2L\left\{\int_0^1 (x-\tfrac{1}{2})\,(S(x))^{\frac{1}{2}}\,⨍_0^1\frac{S'(\xi)\,d\xi}{x-\xi}\,dx + O(\epsilon)\right\}. \quad (4.5)$$

In terms of the fixed axes $OXYZ$, related to $O'xyz$ by (2.1),

$$\phi(x,y,z) \equiv \phi\left(\frac{X-Ut}{L},\frac{Y}{L},\frac{Z}{L}\right)$$

and the pressure is given by

$$P = P_\infty - \rho\frac{\partial\phi}{\partial t} - \tfrac{1}{2}\rho\left[\left(\frac{\partial\phi}{\partial X}\right)^2 + \left(\frac{\partial\phi}{\partial Y}\right)^2 + \left(\frac{\partial\phi}{\partial Z}\right)^2\right]. \tag{4.6}$$

At the bow and stern, $\phi = U(X-Ut)$ and (4.2) is readily recovered from (4.6). Further, in (4.3) the $O(\epsilon)$ term arises, when $S(x)$ is not small, from the term $(\rho U/L)\,\partial\phi/\partial x$ in (4.1), i.e. the term $-\rho\,\partial\phi/\partial t$ in (4.6).

5. Two vessels in parallel motion

Suppose now that there are two vessels moving at different speeds along parallel courses which are a distance D apart, where $D = O(L)$, and introduce notation corresponding to that defined in § 2 as follows:

$$L_j x_j = X - U_j t,\quad L_j y_j = Y - \tfrac{1}{2}(-1)^j D,\quad L_j z_j = Z \quad (j = 1,2),$$

$$\epsilon_j L_j = h \quad (j = 1,2),$$

$$\mathscr{B}_j\colon r_j = (y_j^2 + z_j^2)^{\frac{1}{2}} = \epsilon_j (S_j(x_j))^{\frac{1}{2}} \quad (0 < x_j < 1) \quad (j = 1,2),$$

where

$$S_j(x) = \sum_{n=1}^{\infty} c_{nj} x_j^n \quad \text{near} \quad x_j = 0; \quad S_j(x) = \sum_{n=1}^{\infty} d_{nj}(1-x_j)^n \quad \text{near} \quad x_j = 1.$$

Note that, since $U_1 \neq U_2$, the origin of time can be chosen to be the instant when the sterns are alongside. The total velocity potential ϕ is given by

$$\phi = \phi_1 + \phi_2 + \phi_{12} + \phi_{21}, \tag{5.1}$$

where ϕ_j $(j = 1, 2)$ is the potential due to the motion of the jth vessel in the absence of the other and ϕ_{12}, ϕ_{21} are interaction potentials. Evidently (2.11) and subsequent results imply that

$$\phi_j = U_j L_j \sum_{p=0}^{\infty} \frac{\Gamma(2p+\frac{1}{2})\,\epsilon_j^{2p+2}}{\Gamma(\frac{1}{2})\,(2p)!} \int_{\alpha_j(\epsilon_j)}^{\beta_j(\epsilon_j)} [\xi - \alpha_j(\epsilon_j)]^{2p}\,[\beta_j(\epsilon_j) - \xi]^{2p}\, g_{2p}^{(j)}(\xi,\epsilon_j) \times G_{2p}(x_j - \xi, y_j, z_j, \epsilon_j)\,d\xi, \tag{5.2}$$

where

$$g_{2p}^{(j)}(x_j,\epsilon_j) \simeq g_{2p}^{(j)}(x_j) + O(\epsilon_j^2 \ln \epsilon_j) \quad (p \geqslant 0), \quad g_0^{(j)}(x_j) = \tfrac{1}{4} S_j'(x_j). \tag{5.3}$$

The determination of the interaction potentials must begin by considering the behaviour of ϕ_j near $\mathscr{B}_k$, where $(j,k) = (1,2)$ or $(2,1)$ here and below. For points in the 'outer field', (5.2) and (5.3) yield

$$\phi_j(x_j,y_j,z_j) \simeq \phi_j(x_j,y_j,0) \simeq U_j L_j \epsilon_j^2 \int_0^1 g_0^{(j)}(\xi)\, G_0(x_j-\xi, y_j, 0, \epsilon_j)\,d\xi$$

$$= -\tfrac{1}{4} U_j h \epsilon_j \int_0^1 S_j(\xi) \sum_{m=-\infty}^{\infty} \frac{x_j - \xi}{[(x_j-\xi)^2 + y_j^2 + 4m^2\epsilon_j^2]^{\frac{3}{2}}}\,d\xi$$

after integrating by parts. Then

$$\phi_j(x_j,y_j,z_j) \simeq -\tfrac{1}{4} U_j h \int_0^1 S_j(\xi)\, \frac{x_j-\xi}{(x_j-\xi)^2 + y_j^2}\,d\xi$$

$$= -\tfrac{1}{8} U_j h \int_0^1 S_j'(\xi) \ln[(x_j-\xi)^2 + y_j^2]\,d\xi \tag{5.4}$$

with relative error $O(\epsilon_j)$. This last expression is of the same form as the first integral in equation (12) of Yeung (1978) and can be identified as the two-dimensional potential due to a distribution of line sources of density $-\frac{1}{4}\pi U_j h S_j'(x_j)$ on the axis $0 < x_j < 1$, $y_j = 0$ of the jth vessel. The cross-flow velocity due to (5.4) is given by

$$\frac{1}{L_j}\frac{\partial \phi_j}{\partial y_j} \simeq -\frac{U_j h}{4L_j}\int_0^1 \frac{S_j'(\xi)\, y_j\, d\xi}{(x_j-\xi)^2+y_j^2}$$

and its values V_{kj} on the axis $0 < x_k < 1, y_k = 0$ of the kth vessel, where $L_j y_j = (-1)^k D$, $L_j x_j = L_k x_k + (U_k - U_j)\,t$, are, with relative error $O(\epsilon_j)$, equal to

$$V_{kj}(x_k,t) = \frac{(-1)^{k+1}}{4} U_j h D \int_0^1 \frac{S_j'(\xi)\, d\xi}{[L_k x_k + (U_k-U_j)\,t - L_j\xi]^2 + D^2}. \tag{5.5}$$

The normal derivative of ϕ_j at the surface $\mathscr{B}_k$ is given by

$$\begin{aligned}\left(\frac{\partial \phi_j}{\partial n}\right)_{\mathscr{B}_k} &= \frac{2(S_k(x_k))^{\frac{1}{2}}}{[\epsilon_k^2[S_k'(x_k)]^2+4S_k(x_k)]^{\frac{1}{2}}}\left\{\cos\theta\frac{\partial\phi_j}{\partial y_k}+\sin\theta\frac{\partial\phi_j}{\partial z_k}-\frac{\epsilon_k S_k'(x_k)}{2(S_k(x_k))^{\frac{1}{2}}}\frac{\partial\phi_j}{\partial x_k}\right\}_{r_k\,=\,\epsilon_k(S_k(x_k))^{\frac{1}{2}}} \\ &\simeq \frac{2(S_k(x_k))^{\frac{1}{2}} L_k\cos\theta}{[\epsilon_k^2[S_k'(x_k)]^2+4S_k(x_k)]^{\frac{1}{2}}} V_{kj}(x_k,t)\end{aligned} \tag{5.6}$$

after substitution of (5.4) and (5.5). The contribution of the x_k-derivative in $(\partial\phi_j/\partial n)_{\mathscr{B}_k}$ becomes significant when $x_k(1-x_k) = O(\epsilon_k^2)$ but is omitted from (5.6) because it cannot affect the terms retained in the subsequent calculation. Thus in any plane x_k = constant ($0 < x_k < 1$) the dominant contribution to (5.6) at a given instant of time is that due to the cross-flow velocity defined by (5.5).

Now since (5.2) is the exact solution for ϕ_j in the absence of vessel k, equation (5.1) implies that the boundary condition at $\mathscr{B}_k$ is

$$\frac{\partial}{\partial n}(\phi_j+\phi_{kj}+\phi_{jk}) = 0 \quad \text{at } \mathscr{B}_k. \tag{5.7}$$

Then, on writing down a solution of the form (2.11) for ϕ_{kj}, it is seen that the $\cos\theta$ factor in (5.6) implies that, to leading order, only the odd coefficients are non-zero, yielding

$$\begin{aligned}\phi_{kj} \simeq L_k \sum_{p=1}^{\infty} \frac{\Gamma(2p-\frac{1}{2})\,\epsilon_k^{2p}}{\Gamma(\frac{1}{2})\,(2p-1)!}\int_{\alpha_k(\epsilon_k)}^{\beta_k(\epsilon_k)} &[\xi-\alpha_k(\epsilon_k)]^{2p-1}[\beta_k(\epsilon_k)-\xi]^{2p-1} \\ &\times f_{2p-1}^{(k)}(\xi,t,\epsilon_k)\, G_{2p-1}(x_k-\xi, y_k, z_k, \epsilon_k)\, d\xi.\end{aligned} \tag{5.8}$$

Evidently the leading terms $\{f_{2p-1}^{(k)}(\xi,t)\}$ of $\{f_{2p-1}^{(k)}(\xi,t,\epsilon_k)\}$ respectively are evaluated without any contribution from ϕ_{jk} to (5.7), i.e. these leading terms correspond to application of the approximate condition

$$\frac{\partial}{\partial n}(\phi_j+\phi_{kj}) = 0 \quad \text{on} \quad \mathscr{B}_k$$

in which the presence of $\mathscr{B}_j$ is ignored and ϕ_{kj} is required to cancel the normal derivative of ϕ_j on $\mathscr{B}_k$. The inclusion of ϕ_{jk} in (5.7) enables $f_1^{(j)}(\xi,t)$ to influence the even coefficients in ϕ_{kj}, corresponding to a higher-order interaction. Such calculations require consideration of the errors ignored in (5.4) and (5.6) and will not be pursued here.

Continuing as in §3, the substitution of (5.6) and (5.8) into condition (5.7) shows that the functions $\{f_{2p-1}^{(k)}(x,t);\, p \geqslant 1\}$ are determined by the equation

$$\sum_{p=1}^{\infty} [x(1-x)]^{2p-1} f_{2p-1}^{(k)}(x,t) \left\{ \frac{\cos(2p-1)\theta}{[S_k(x)]^p} + \mathscr{R} \sum_{m=1}^{\infty} \left(\frac{1}{Q_{m,k}^{2p}} + \frac{1}{Q_{-m,k}^{2p}} \right) e^{-i\theta} \right\} = V_{kj}(x,t)\cos\theta \quad (-\pi \leqslant \theta \leqslant \pi,\ 0 < x < 1) \tag{5.9}$$

where $Q_{m,k} = (S_k(x))^{\frac{1}{2}} e^{-i\theta} - 2mi$ and V_{kj} is given by (5.5). The left-hand side of (5.9) has the Fourier series

$$\sum_{p=1}^{\infty} [x(1-x)]^{2p-1} f_{2p-1}^{(k)}(x,t) \left\{ \frac{\cos(2p-1)\theta}{[S_k(x)]^p} + \sum_{s=p}^{\infty} \frac{(-1)^s}{2s-1} [S_k(x)]^{s-p} \times \frac{(2s-1)!\,\zeta(2s)}{(2p-1)!\,(2s-2p)!} \cos(2s-2p+1)\theta \right\},$$

whence, on equating Fourier coefficients in (5.9), it follows that

$$\frac{x(1-x)}{S_k(x)} f_1^{(k)}(x,t) + \sum_{p=1}^{\infty} f_{2p-1}^{(k)}(x,t) \left[\frac{x(1-x)}{2} \right]^{2p-1} (-1)^p \zeta(2p) = V_{kj}(x,t), \tag{5.10a}$$

$$\left[\frac{x(1-x)}{S_k(x)} \right]^{2q+1} f_{2q+1}^{(k)}(x,t) + \sum_{p=1}^{\infty} \frac{[x(1-x)]^{2p-1} f_{2p-1}^{(k)}(x,t)\,(-1)^{p+q}(2p+2q-1)!\,\zeta(2p+2q)}{2^{2p+2q-1}(2p-1)!\,(2q)!} = 0 \quad (q \geqslant 1). \tag{5.10b}$$

Remarkably, it is still possible to write down a solution for $\{f_{2q+1}^{(k)}(x,t);\, q \geqslant 0\}$ in terms of an expansion in powers of $S_k(x)$ in which the coefficients are purely numerical. Thus

$$f_1^{(k)}(x,t) = \frac{S_k(x)}{x(1-x)} [1 + B_0^{(k)}(x)] V_{kj}(x,t), \tag{5.11a}$$

$$f_{2q+1}^{(k)}(x,t) = \left[\frac{S_k(x)}{x(1-x)} \right]^{2q+1} B_q^{(k)} V_{kj}(x,t) \quad (q \geqslant 1), \tag{5.11b}$$

and the equations determining recursively the coefficients $\{b_{qr};\, q \geqslant 0, r \geqslant 1\}$ in the expansions

$$B_q^{(k)}(x) = (-1)^q \sum_{r=1}^{\infty} b_{qr} [S_k(x)]^r \quad (q \geqslant 0) \tag{5.12}$$

are obtained by substituting for $f_{2q+1}^{(k)} (q \geqslant 0)$ in (5.10a,b) and equating powers of $S_k(x)$. Hence

$$b_{q1} = \frac{(2q+1)}{2^{2q+1}} \zeta(2+2q) \quad (q \geqslant 0), \tag{5.13}$$

$$(b_{q,2n};\, b_{q,2n+1}) = \sum_{p=1}^{n} \frac{(2p+2q-1)!\,\zeta(2p+2q)}{(2p-1)!\,(2q)!\,2^{2p+2q-1}} (b_{p-1,2n-2p+1};\, b_{p-1,2n-2p+2}) \quad (q \geqslant 0,\quad n \geqslant 1).$$

Note that, in contrast to the single-vessel case, both odd and even powers of S_k are required in the expansions. Evidently the coefficients can be calculated sequentially column by column, with each element in the $\begin{pmatrix} 2n \\ 2n+1 \end{pmatrix}$th column ($n \geqslant 1$) being a linear combination of the n elements which lie on the counter diagonal at the top of the matrix formed by the previous n $\binom{\text{odd}}{\text{even}}$ columns. Numerical computation showed that $0 < b_{qr} < 1$ for all $q \geqslant 0$, $r \geqslant 1$ and both the ratio and root tests indicated convergence of each series (5.12) for all $S_k(x)$ such that $0 < S_k(x) < 1$, as required. Thus the potential ϕ for two vessels moving on parallel courses in shallow water is determined by equations (5.1)–(5.3), (5.8) and (5.11)–(5.13).

For further comparison with equation (12) of Yeung (1978), the behaviour of expression (5.8) in the 'outer field' of vessel k is evidently

$$\begin{aligned}\phi_{kj}(x_k, y_k, z_k, t) &\simeq \tfrac{1}{8}\epsilon_k^2 L_k \int_0^1 \xi(1-\xi) f_1^{(k)}(\xi, t)\, G_1(x_k-\xi, y_k, 0, \epsilon_k)\, d\xi \\ &= \tfrac{1}{8}h\epsilon_k \int_0^1 \xi(1-\xi) f_1^{(k)}(\xi, t) \sum_{m=-\infty}^{\infty} \frac{y_k}{[(x_k-\xi)^2 + y_k^2 + 4m^2\epsilon_k^2]^{\frac{3}{2}}}\, d\xi \\ &\simeq \tfrac{1}{4}h \int_0^1 \xi(1-\xi) f_1^{(k)}(\xi, t) \frac{y_k}{(x_k-\xi)^2 + y_k^2}\, d\xi \\ &= -\tfrac{1}{4}h \int_0^1 \tan^{-1}\left(\frac{y_k}{x_k-\xi}\right) \frac{d}{d\xi}[\xi(1-\xi) f_1^{(k)}(\xi, t)]\, d\xi. \end{aligned} \tag{5.14}$$

This expression, which is the two-dimensional potential due to a distribution of vortices along the line $y_k = 0$, $0 < x_k < 1$, is continuous across the branch cut of the arctangent function because the total vorticity is zero. Hence no trailing vortices are generated in the present case, characterized by $S_k'(x) \neq 0$ at $x = 0, 1$.

6. The hydrodynamic forces on the vessels

The fluid pressure can be found from (4.6) after writing the expression (5.1) in terms of X, Y, Z and t. In particular, at the surface $\mathscr{B}_k$, except where S_k is small,

$$\frac{1}{\rho}(P - P_\infty)_{\mathscr{B}_k} \simeq -\left[\frac{\partial}{\partial t}(\phi_k + \phi_j + \phi_{kj})\right]_{\mathscr{B}_k} + O(\epsilon_1\epsilon_2 U_1 U_2) \tag{6.1}$$

because the velocities are then uniformly of order ϵ_k and ϕ_{jk} is of order $U_k h^2/D$ in the 'far field'. Now, by comparison with (4.3),

$$\left(\frac{\partial\phi_k}{\partial t}\right)_{\mathscr{B}_k} \simeq \tfrac{1}{4}U_k^2\epsilon_k \int_0^1 \frac{S_k'(\xi)\, d\xi}{x_k-\xi} \quad (0 < x_k < 1)$$

whilst (5.4) implies that, near $\mathscr{B}_k$,

$$\phi_j \simeq -\tfrac{1}{8}U_j h \int_0^1 S_j'(\xi) \ln[(X - U_j t - L_j\xi)^2 + (Y - \tfrac{1}{2}(-1)^j D)^2]\, d\xi.$$

Hence,

$$\left[\frac{\partial}{\partial t}\phi_j(X, Y, Z, t)\right]_{\mathscr{B}_k} \simeq \tfrac{1}{4}U_j^2 h \int_0^1 \frac{S_j'(\xi)\,[L_k x_k + (U_k - U_j)t - L_j\xi]}{[L_k x_k + (U_k - U_j)t - L_j\xi]^2 + D^2}\, d\xi \quad (0 < x_k < 1). \tag{6.2}$$

Thus the dominant contribution to the pressure at $\mathscr{B}_k$ due to the motion of vessel j is independent of θ and therefore cannot produce any lateral force. Further (5.8) implies that

$$
\begin{aligned}
(\phi_{kj})_{\mathscr{B}_k} &\simeq L_k \sum_{p=1}^{\infty} \frac{\Gamma(2p-\frac{1}{2})\,\epsilon_k^{2p}}{\Gamma(\frac{1}{2})\,(2p-1)!} \int_{\alpha_k(\epsilon_k)}^{\beta_k(\epsilon_k)} [\xi-\alpha_k(\epsilon_k)]^{2p-1}[\beta_k(\epsilon_k)-\xi]^{2p-1} \\
&\quad \times f_{2p-1}^{(k)}(\xi,t,\epsilon_k)\,\mathscr{R} \sum_{m=-\infty}^{\infty} \frac{(\epsilon_k \overline{Q_{m,k}})^{2p-1}}{[(x_k-\xi)^2+\epsilon_k^2|Q_{m,k}|^2]^{2p-\frac{1}{2}}}\,d\xi \\
&\simeq h \sum_{p=1}^{\infty} \frac{[x_k(1-x_k)]^{2p-1}}{2p-1} f_{2p-1}^{(k)}(x_k,t)\,\mathscr{R} \sum_{m=-\infty}^{\infty} \frac{1}{Q_{m,k}^{\;2p-1}}
\end{aligned} \tag{6.3}
$$

after using the methods of § 3. Since the small parameter appears here only in the factor h^2, it is clear that, after writing x_k in terms of X, t, the contribution of ϕ_{kj} to (6.1) must be of second order.

This simple analysis, using the fixed axes $OXYZ$, shows how the sinkage force is, to leading order, modified by the second vessel and that a higher-order calculation is required to determine the dominant terms in the lateral forces. The modification of equations (4.4) and (4.5) is that the downward force $F_S^{(k)}$ and the trimming moment $M_T^{(k)}$ about the line $x_k = \frac{1}{2}$, $z = 0$, which are exerted by the fluid on the 'wetted' half of the surface $\mathscr{B}_k$ due to the motion of both vessels, are given by

$$
\begin{aligned}
\begin{pmatrix} F_S^{(k)} \\ M_T^{(k)} \end{pmatrix} &\simeq \tfrac{1}{4}\rho h^2 \int_0^1 (S_k(x))^{\frac{1}{2}} \begin{pmatrix} 1 \\ L_k(\frac{1}{2}-x) \end{pmatrix} \left\{ U_k^2 \int_0^1 \frac{S_k'(\xi)\,d\xi}{x-\xi} \right. \\
&\quad \left. + U_j^2 L_k \int_0^1 \frac{[L_k x + (U_k-U_j)\,t - L_j\xi]\,S_j'(\xi)}{[L_k x + (U_k-U_j)\,t - L_j\xi]^2 + D^2}\,d\xi \right\} dx.
\end{aligned} \tag{6.4}
$$

Evidently the steady case of two ships moving with equal velocity U can be recovered from formula (6.4) by setting $U_1 = U = U_2$ and replacing $(U_k - U_j)\,t$ by the constant distance by which the stern of vessel k is longitudinally in advance of the stern of vessel j.

Since the velocities are no longer negligible, the second-order calculation, required for the lateral forces, is simplified by using axes fixed in vessel k in order to find the pressure at $\mathscr{B}_k$. In contrast to (4.1), there remains a time derivative, with P given by

$$
\frac{P-P_\infty}{\rho} = -\frac{\partial}{\partial t}\phi(x_k,y_k,z_k,t) + \tfrac{1}{2}U_k^2\left[1 - \left\{\operatorname{grad}\left[\frac{\phi(x_k,y_k,z_k,t)}{U_k L_k} - x_k\right]\right\}^2\right]. \tag{6.5}
$$

The immediate simplifications here are that $\phi_k(x_k,y_k,z_k,t) \equiv \phi_k(x_k,y_k,z_k)$, given by (5.2), and the boundary conditions at $\mathscr{B}_k$ imply that the normal component of the gradient is equal to $(U_k L_k)^{-1}(\partial\phi_{jk}/\partial n)_{\mathscr{B}_k} = O(\epsilon_j \epsilon_k)$ and therefore negligible.

Consider first the remaining terms in $\partial\phi/\partial t$. Equation (5.4) implies that, near $\mathscr{B}_k$,

$$
\begin{aligned}
\phi_j(x_k,y_k,z_k,t) &\simeq -\tfrac{1}{8}U_j h \int_0^1 S_j'(\xi) \ln\{[L_k x_k + (U_k-U_j)\,t - L_j\xi]^2 + [(-1)^k D + L_k y_k]^2\}\,d\xi \\
&\simeq -\tfrac{1}{8}U_j h \int_0^1 S_j'(\xi) \ln\{[L_k x_k + (U_k-U_j)\,t - L_j\xi]^2 + D^2\}\,d\xi + L_k y_k V_{kj}(x_k,t),
\end{aligned} \tag{6.6}
$$

where V_{kj} is the cross-flow velocity defined by (5.5) and y_k has been assumed small. In particular

$$
\begin{aligned}
[\phi_j(x_k,y_k,z_k,t)]_{\mathscr{B}_k} &\simeq -\tfrac{1}{8}U_j h \int_0^1 S_j'(\xi) \ln\{[L_k x_k + (U_k-U_j)\,t - L_j\xi]^2 + D^2\}\,d\xi \\
&\quad + h(S_k(x_k))^{\frac{1}{2}} V_{kj}(x_k,t)\cos\theta.
\end{aligned}
$$

Also substitution of the Fourier series

$$\mathscr{R} \sum_{m=-\infty}^{\infty} \frac{1}{Q_{m,k}^{2p-1}} = \frac{\cos(2p-1)\theta}{[S_k(x_k)]^{p-\frac{1}{2}}} + \sum_{s=p}^{\infty} \frac{(-1)^{s-1}[S_k(x_k)]^{s-p+\frac{1}{2}}(2s-1)!\,\zeta(2s)}{2^{2s-1}(2p-2)!\,(2s-2p+1)!}\cos(2s-2p+1)\theta$$

into (6.3) yields

$$(\phi_{kj})_{\mathscr{B}_k} \simeq h\sum_{p=1}^{\infty} \frac{[x_k(1-x_k)]^{2p-1}}{2p-1} f_{2p-1}^{(k)}(x_k,t)\left\{\frac{\cos(2p-1)\theta}{[S_k(x_k)]^{p-\frac{1}{2}}}\right.$$

$$\left. + \sum_{q=0}^{\infty} \frac{(-1)^{p+q-1}[S_k(x_k)]^{q+\frac{1}{2}}(2p-2q-1)!\,\zeta(2p+2q)}{2^{2p+2q-1}(2p-2)!\,(2q+1)!}\cos(2q+1)\theta\right.$$

$$= 2h\sum_{q=0}^{\infty} \frac{[x_k(1-x_k)]^{2q+1} f_{2q+1}^{(k)}(x_k,t)}{[S_k(x_k)]^{q+\frac{1}{2}}(2q+1)}\cos(2q+1)\theta - h(S_k(x_k))^{\frac{1}{2}}V_{kj}(x_k,t)\cos\theta$$

after using equations (5.10*a*, *b*) to simplify the double summations. It is this rearrangement which ensures that only $f_1^{(k)}$ contributes to the leading terms of the sway force. Equation (5.14) shows that this was effectively anticipated by Yeung (1978) in his formation of the 'outer' solution. Then, on combining $(\phi_{kj})_{\mathscr{B}_k}$ with $(\phi_j)_{\mathscr{B}_k}$ and substituting (5.11*a*, *b*), it follows that

$$(\phi_j+\phi_{kj})_{\mathscr{B}_k} \simeq -\tfrac{1}{8}U_j h\int_0^1 S_j'(\xi)\ln\{[L_k x_k + (U_k-U_j)t - L_j\xi]^2 + D^2\}\,d\xi$$
$$+ 2hV_{kj}(x_k,t)(S_k(x_k))^{\frac{1}{2}}\left\{\cos\theta + \sum_{q=0}^{\infty} B_q^{(k)}(x_k)\frac{[S_k(x_k)]^q}{2q+1}\cos(2q+1)\theta\right\}. \tag{6.7}$$

Further, near $\mathscr{B}_k$,

$$\phi_{jk} \simeq \tfrac{1}{2}h\int_0^1 \xi(1-\xi)f_1^{(j)}(\xi,t)\frac{y_j\,d\xi}{(x_j-\xi)^2+y_j^2}$$
$$= \tfrac{1}{2}h\int_0^1 \frac{[L_k y_k + (-1)^k D]\,\xi(1-\xi)f_1^{(j)}(\xi,t)\,d\xi}{[L_k x_k + (U_k-U_j)t - L_j\xi]^2 + [L_k y_k + (-1)^k D]^2}$$

and hence, since $f_1^{(j)}$ is of the same order of magnitude as V_{jk}, the contributions of ϕ_{jk} to all terms in (6.5) are an order of magnitude smaller than the corresponding contributions of ϕ_j.

Consider next the tangential components of $\{\mathrm{grad}\,[(U_k L_k)^{-1}\phi(x_k,y_k,z_k,t) - x_k]\}_{\mathscr{B}_k}$. By comparison with § 4, the first tangential component of

$$\{\mathrm{grad}\,[(U_k L_k)^{-1}(\phi_k+\phi_{kj}) - x_k]\}_{\mathscr{B}_k}$$

is asymptotically

$$\frac{2(S_k(x_k))^{\frac{1}{2}}}{\{\epsilon_k^2[S_k'(x_k)]^2 + 4S_k(x_k)\}^{\frac{1}{2}}}\left\{-1 - \tfrac{1}{4}\epsilon_k \int_0^1 \frac{S_k'(\xi)\,d\xi}{x_k-\xi} + \epsilon_k^2 \times (\text{even function of } \cos\theta) + (U_k L_k)^{-1}\frac{\partial}{\partial x_k}(\phi_{kj})_{\mathscr{B}_k}\right\},$$

whilst the second tangential component, to order ϵ_k, has a Fourier series involving only $\{\sin q\theta;\ q \geqslant 1\}$.

The first tangential component of $\{\text{grad}\,[(U_k L_k)^{-1}(\phi_j+\phi_{jk})]\}_{\mathscr{B}_k}$ is given by

$$\frac{2(S_k(x_k))^{\frac{1}{2}}}{\{\epsilon_k^2[S_k'(x_k)]^2+4S_k(x_k)\}^{\frac{1}{2}}}\left\{\left[\frac{\partial}{\partial x_k}+\frac{\epsilon_k S_k'(x_k)}{2(S_k(x_k))^{\frac{1}{2}}}\left(\cos\theta\frac{\partial}{\partial y_k}+\sin\theta\frac{\partial}{\partial z_k}\right)\right]\frac{(\phi_j+\phi_{jk})}{U_k L_k}\right\}_{\mathscr{B}_k}$$

$$\simeq \frac{2(S_k(x_k))^{\frac{1}{2}}}{\{\epsilon_k^2[S_k'(x_k)]^2+4S_k(x_k)\}^{\frac{1}{2}}}\frac{1}{U_k L_k}\left\{\frac{\partial\phi_j}{\partial x_k}+\frac{\epsilon_k S_k'(x_k)}{2(S_k(x_k))^{\frac{1}{2}}}\cos\theta\frac{\partial\phi_j}{\partial y_k}\right\}_{\mathscr{B}_k}$$

$$\simeq \frac{2(S_k(x_k))^{\frac{1}{2}}}{\{\epsilon_k^2[S_k'(x_k)]^2+4S_k(x_k)\}^{\frac{1}{2}}}\frac{1}{U_k L_k}\frac{\partial}{\partial x_k}(\phi_j)_{\mathscr{B}_k}$$

by substitution of (6.6). The second tangential component is given by

$$\left\{\left(-\sin\theta\frac{\partial}{\partial y_k}+\cos\theta\frac{\partial}{\partial z_k}\right)\frac{(\phi_j+\phi_{jk})}{U_k L_k}\right\}_{\mathscr{B}_k}\simeq -\frac{1}{U_k}V_{kj}(x_k,t)\sin\theta.$$

Hence, on substitution of these results in (6.5), the pressure at the surface $\mathscr{B}_k$ is given, to second order, by

$$\frac{1}{\rho}(P-P_\infty)_{\mathscr{B}_k}\simeq -\frac{\partial}{\partial t}(\phi_j+\phi_{kj})_{\mathscr{B}_k}+\frac{4S_k(x_k)}{\epsilon_k^2[S_k'(x_k)]^2+4S_k(x_k)}$$
$$\times\left\{\frac{U_k}{L_k}\frac{\partial}{\partial x_k}(\phi_j+\phi_{kj})_{\mathscr{B}_k}+(\text{even function of }\cos\theta)\right\}. \quad (6.8)$$

This expression takes full account of the end effects as did the corresponding formula (4.3) for the single vessel. The desire for uniform accuracy was the reason for precluding the apparently simpler procedure of writing (6.5) in the form

$$\frac{P-P_\infty}{\rho}=-\left(\frac{\partial}{\partial t}-\frac{U_k}{L_k}\frac{\partial}{\partial x_k}\right)\phi(x_k,y_k,z_k,t)-\frac{1}{2L_k^2}\left[\left(\frac{\partial\phi}{\partial x_k}\right)^2+\left(\frac{\partial\phi}{\partial y_k}\right)^2+\left(\frac{\partial\phi}{\partial z_k}\right)^2\right]$$

and seeking to show that the dominant terms are those involving ϕ_j and ϕ_{kj} which arise from the time derivative in the moving frame. The steady situation is recovered from (6.8) by writing $U_1=U=U_2$ and replacing $(U_1-U_2)t$ by a constant distance of separation as described earlier.

Evidently the term

$$2hV_{kj}(x_k,\epsilon)\,[1+B_0^{(k)}(x_k)]\,(S_k(x_k))^{\frac{1}{2}}\cos\theta$$

in (6.7) corresponds to the term $V_j^*\Phi_j^{(2)}$ in equation (18) of Yeung (1978) because when it is written in the form

$$L_k V_{kj}(x_k,t)\,[1+B_0^{(k)}(x_k)]\left\{y_k\left[1+\frac{\epsilon_k^2 S_k(x_k)}{r_k^2}\right]\right\}_{\mathscr{B}_k}$$

it is readily identified as the values on $\mathscr{B}_k$ of the potential due to a lateral cross-flow with velocity $V_{kj}(1+B_0^{(k)})$ past the cylinder $y_k^2+z_k^2=\epsilon_k^2 S_k$. Thus the function $B_0^{(k)}$, which depends only on the shape of $\mathscr{B}_k$, measures how the interactions between the ships modify the observed cross-flow velocity. Although $B_0^{(k)}$ is given by (5.11*a*), it must be remembered that its determination involves the higher-order singularities. However the structure of equations (5.13) ensures that this is a simple recursive calculation.

Since the lateral force $F_L^{(k)}$ exerted by the fluid on the 'wetted' half of vessel k in the direction of Y increasing is given by

$$F_L^{(k)}=-hL_k\int_0^1\int_0^\pi (P-P_\infty)_{\mathscr{B}_k}(S_k(x_k))^{\frac{1}{2}}\cos\theta\,d\theta\,dx_k, \quad (6.9)$$

the only terms on the right-hand side of (6.8) which make non-zero contributions to (6.9) are, on substituting (6.7),

$$-2h\cos\theta\left(\frac{\partial}{\partial t}-\frac{4S_k(x_k)}{\epsilon_k{}^2[S_k'(x_k)]^2+4S_k(x_k)}\frac{U_k}{L_k}\frac{\partial}{\partial x_j}\right)\{V_{kj}(x_k,t)\,[1+B_0^{(k)}(x_k)]\,(S_k(x_k))^{\frac{1}{2}}\}.$$

However, when these are inserted into (6.9), the end effects are immaterial to the leading term of $F_L^{(k)}$, leaving

$$F_L^{(k)} \simeq \pi h^2\rho\int_0^1 (S_k(x))^{\frac{1}{2}}\left\{\left(L_k\frac{\partial}{\partial t}-U_k\frac{\partial}{\partial x}\right)V_{kj}(x,t)\,[1+B_0^{(k)}(x)]\,(S_k(x))^{\frac{1}{2}}\right\}dx$$

$$= \pi h^2\rho\int_0^1 [1+B_0^{(k)}(x)]\left\{L_k\frac{\partial V_{kj}}{\partial t}(x,t)\,S_k(x)+\tfrac{1}{2}U_kV_{kj}(x,t)\,S_k'(x)\right\}dx \qquad (6.10)$$

after an integration by parts. The yaw moment $M_y^{(k)}$ about the line $x_k = \frac{1}{2}$, $y_k = 0$ (directed downward) is given by

$$M_y^{(k)} = -hL_k{}^2\int_0^1\int_0^\pi (P-P_\infty)_{\mathscr{B}_k}(x_k-\tfrac{1}{2})\,(S_k(x_k))^{\frac{1}{2}}\cos\theta\,d\theta\,dx_k$$

$$\simeq \pi h^2L_k\,\rho\int_0^1 (S_k(x))^{\frac{1}{2}}\,(x-\tfrac{1}{2})\left\{\left(L_k\frac{\partial}{\partial t}-U_k\frac{\partial}{\partial x}\right)V_{kj}(x,t)\,[1+B_0^{(k)}(x)]\,(S_k(x))^{\frac{1}{2}}\right\}dx$$

$$= \pi h^2L_k\,\rho\int_0^1 [1+B_0^{(k)}(x)]\left\{(x-\tfrac{1}{2})\,L_k\frac{\partial V_{kj}}{\partial t}(x,t)\,S_k(x)+U_kV_{kj}(x,t)\,S_k(x)\right.$$

$$\left.+\tfrac{1}{2}U_kV_{kj}(x,t)\,(x-\tfrac{1}{2})\,S_k'(x)\right\}dx \qquad (6.11)$$

after similar manipulation.

On comparing (6.10) and (6.11) with equations (52) and (53) of Yeung (1978), with $V_{kj}(1+B_0^{(k)})$ already identified with V_1^* and the cross-sectional areas being $\pi h^2S_k(x)$ and $S_1(x_1)$ in the respective notations, the discrepancy involves the appearance of $2hC_1(x_1)$ instead of $S_1(x_1)$ in some of Yeung's terms. Here $C_1(x_1)$ is the blockage coefficient given by

$$4hC_1(x_1) = A_1(x_1)+S_1(x_1),$$

where A_1 is an added mass coefficient which reduces to S_1 in the absence of external boundaries.

7. Examples and discussion

The formulae (6.10) and (6.11) can be used to calculate efficiently the lateral force and yaw moment exerted on vessels moving on parallel courses in shallow water. They involve only the approximation of the infinite series representation of $B_0^{(1)}(x)$ in (5.12) by evaluating a finite number of the coefficients $\{b_{0,r}\}$ from equations (5.13) and then straightforward numerical quadrature. In each of the examples below, it is found that only 25 terms in (5.12) are needed to ensure less than 1 % error in the results tabulated in the figures. Simpson's rule with 20 panels is used to perform the numerical quadratures required. Further, in all cases, the cross-sectional area of each vessel is taken to be of the form $S_i = C_iS(x)$ $(i = 1,2)$, where the distribution of values of $S(x)$ and the normalization constant C_i are respectively chosen in agreement with the cross-sectional areas and fractional bottom clearances of the vessels quoted by previous authors and with which comparison is being made.

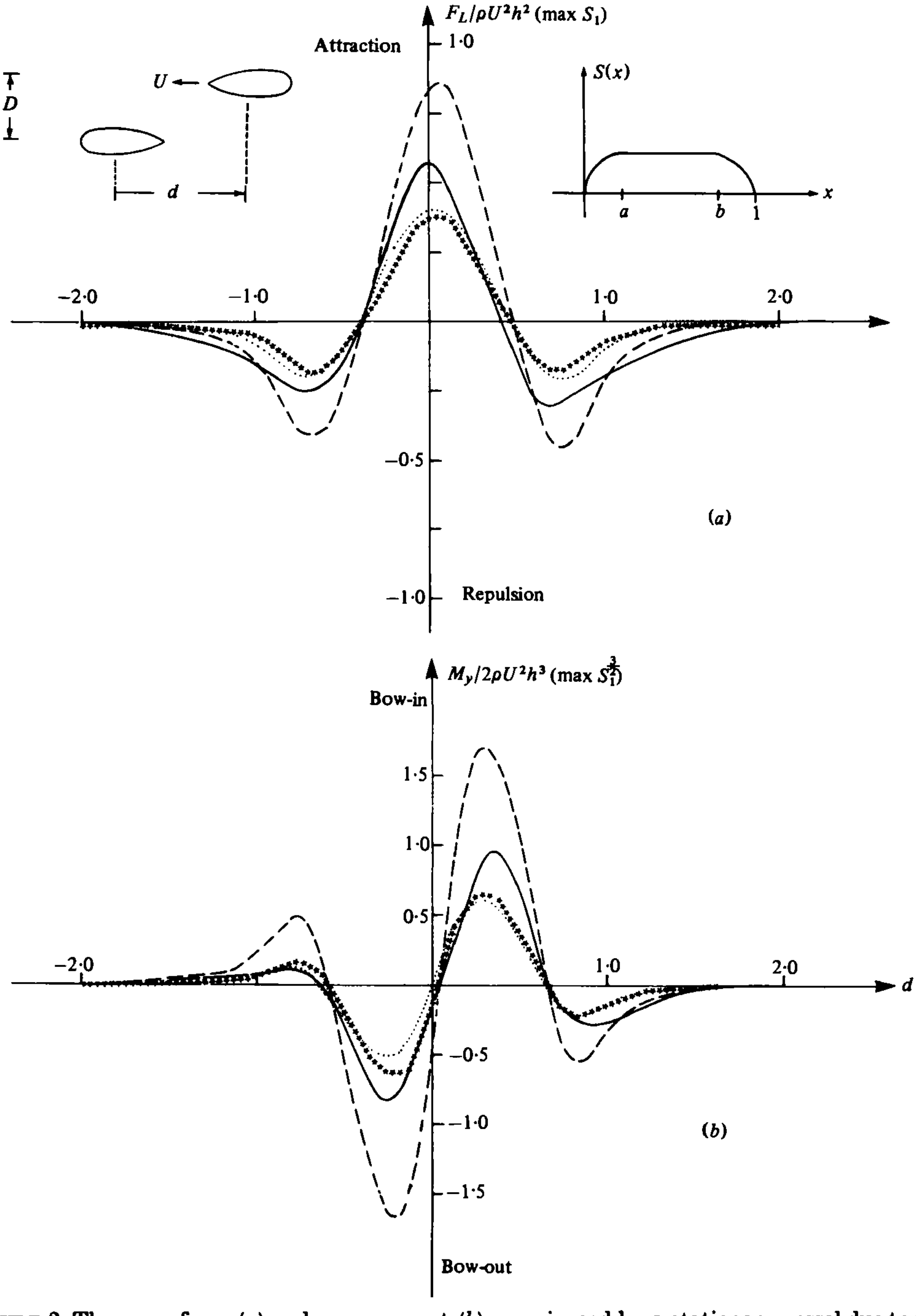

FIGURE 2. The sway force (*a*) and yaw moment (*b*) experienced by a stationary vessel due to the passage of another. The geometrical parameters are $L_2/L_1 = 0{\cdot}712$, $D/L_1 = 0{\cdot}239$, max $S_1 = 0{\cdot}756$, max $S_2 = 0{\cdot}338$, $\epsilon_1 = 0{\cdot}0825$. The experimental data are taken from Remery (1974) and are speed-averaged values with the Froude number varying from 0·155 to 0·270, ——, experiment; – – –, Yeung (unblocked flow); . . . ., Yeung (with blockage); × × × ×, uniform slender body theory.

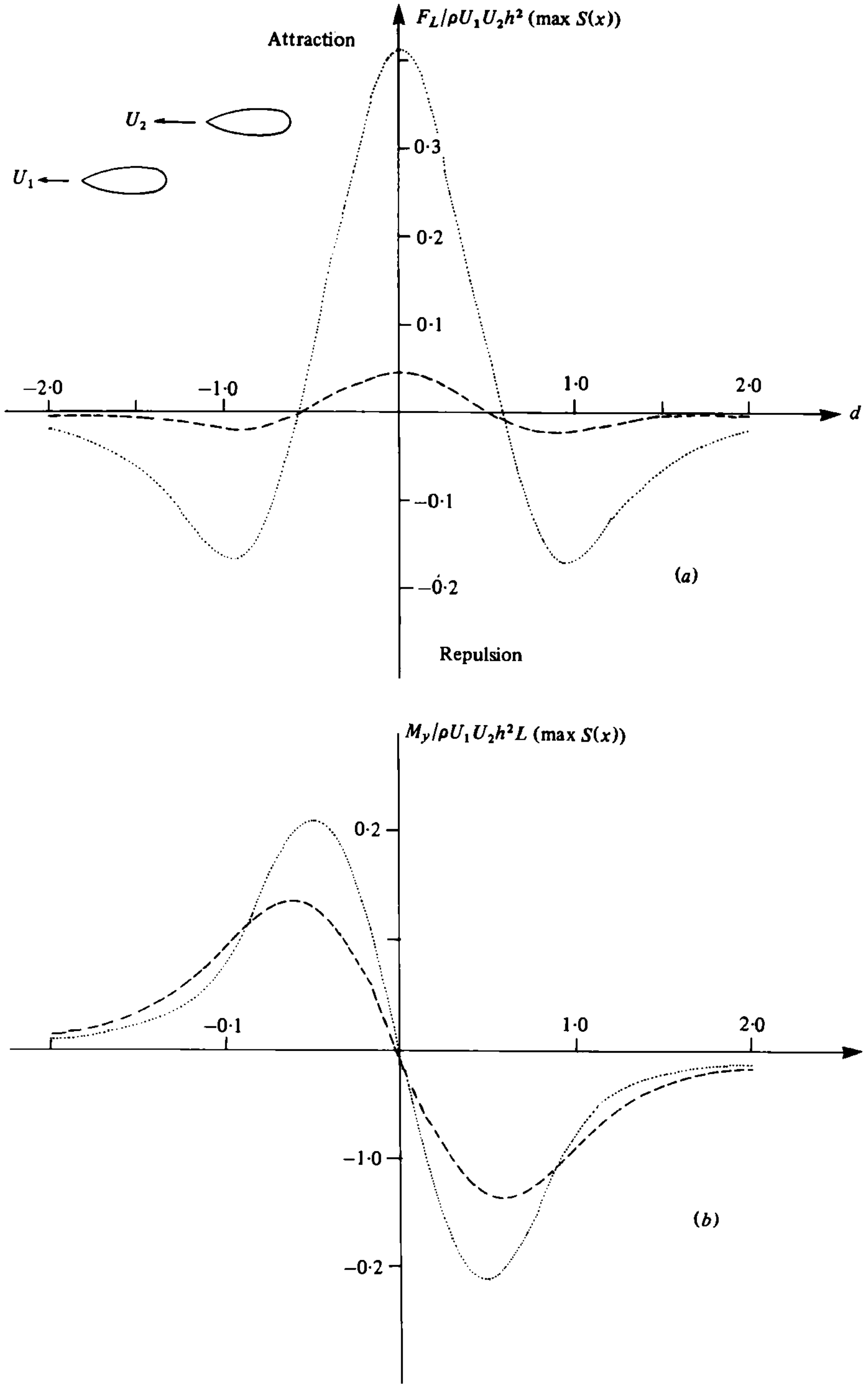

FIGURE 3. The normalized sway force (a) and yaw moment (b) acting on two identical vessels with one overtaking the other for $\epsilon = 0{\cdot}0825$, $D/L = 0{\cdot}5$, $U_2/U_1 = 1{\cdot}5$, and max $S(x) = 0{\cdot}826$. . . . ., slower ship; – – – –, faster ship.

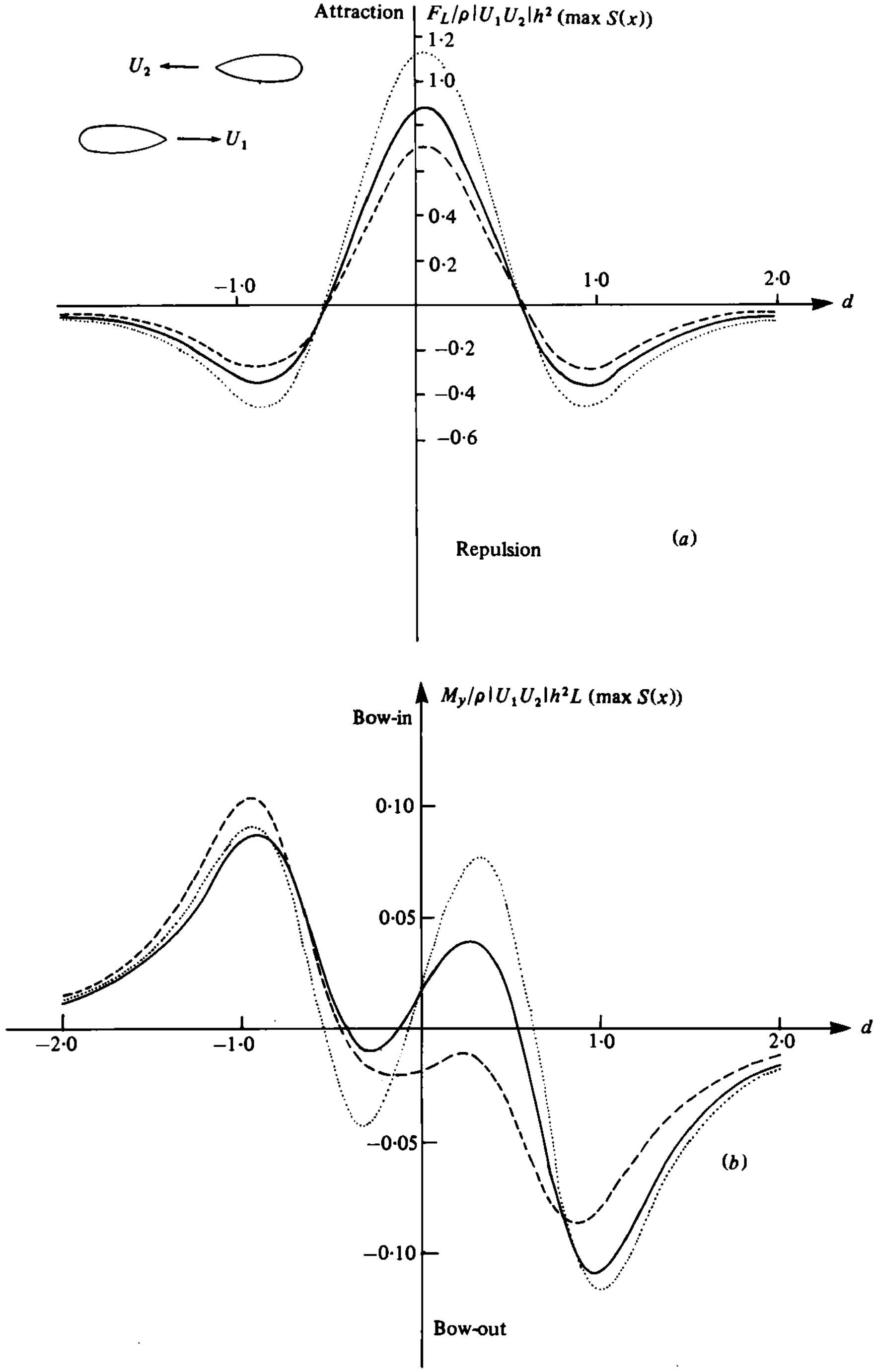

FIGURE 4. The normalized away force (*a*) and yaw moment (*b*) acting on two identical vessels in head-on encounter with $\epsilon = 0{\cdot}0825$, $D/L = 0{\cdot}5$, and max $S(x) = 0{\cdot}826$. ——, $U_2/U_1 = -1{\cdot}0$; ...., $U_2/U_1 = 1{\cdot}5$, slower ship; ----, $U_2/U_1 = 1{\cdot}5$, faster ship.

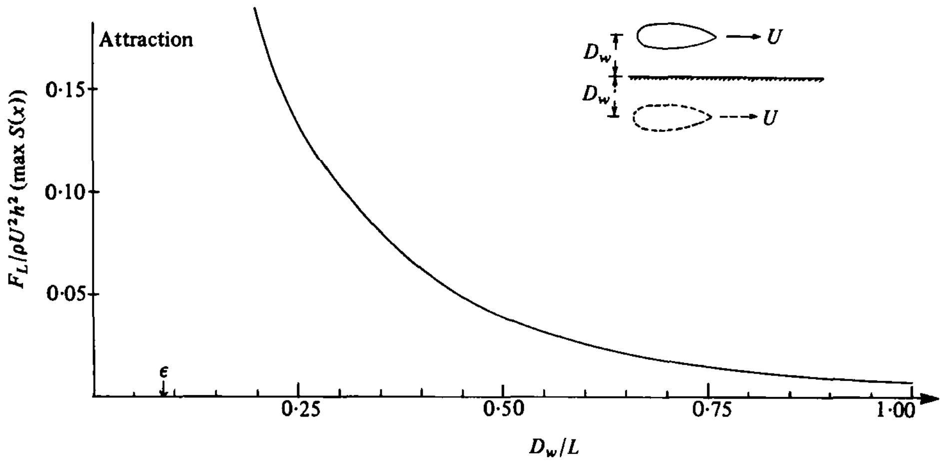

FIGURE 5. The normalized lateral force experienced by a ship moving in shallow water parallel to a fixed vertical wall plotted against the normalized distance D_w/L of the centre-line of the ship from the wall.

In the first example, the sway force and yaw moment experienced by a moored 100k DWT tanker due to the passage of a 30k DWT tanker are computed and the results shown in figures 2 (a, b), together with the theoretical and experimental results displayed by Yeung (1978) in his figure 4. The normalized force and moment are plotted against the normalized stagger distance d, defined as the difference between the X co-ordinates of the centres of the two ships. As can be seen from the plots, the computed theoretical results are close to the more accurate of Yeung's two results. The discrepancy between these and the experimental results may possibly be explained by the marginally acceptable value of D/L which is approximately three times the small parameter ϵ. Furthermore, some allowance for the non-zero Froude number should be made in interpreting the experimental data. The cross-sectional area of each vessel is taken to be such that $S(x)$ consists of a constant mid-section region for $a \leqslant x \leqslant b$, where $0 < a < b < 1$, with parabolic ends (see the insert in figure 2). The results shown in figure 2 correspond to $a = 0{\cdot}3$ and $b = 0{\cdot}8$. It is found that, as the constant mid-section region is lengthened from zero, corresponding to a prolate spheroid, the magnitude of the force and moment increases at all stagger distances. A body with a pointed bow and fin-shaped stern, corresponding to $S(x) = x^3(1-x)^2$, results in a force and moment even smaller than the corresponding values for the prolate spheroid.

The second example involves two identical vessels travelling in the same direction on parallel courses, with one vessel overtaking the other. The normalized lateral force and yaw moment acting on each vessel are shown in figures 3 (a, b). The cross-sectional area is the same as described in the first example, with $a = 0{\cdot}3$ and $b = 0{\cdot}8$. The ratio of separation distance to ship length is 0·5, the ratio of the speeds is 1·5, the beam to length ratio is 0·15, and $\epsilon = 0{\cdot}825$. The bottom clearance is 10 % of the draught of each ship. The slower vessel experiences, as expected, the larger force and moment and, further, its presence is hardly detected by the faster ship.

The third example differs from the second only in that the ships travel in opposite directions. The numerical values of the various parameters involved were deliberately chosen for comparison with Yeung's third example. The normalized lateral force and yaw moment are shown in figures 4 (a, b). Evidently, the curves have similar shape to those of Yeung, but are of uniformly smaller magnitude. This may be reasonably explained by comparison with his figure 4, because his calculations have assumed the flow to be unblocked, i.e. he has ignored the function $B_0^{(k)}$. An additional difference is that the yaw moment on the faster ship experiences only one change of sign.

The final example concerns a ship moving parallel to and at a distance D_w from a vertical wall. The boundary condition ensures that this is equivalent to two identical ships moving side by side a distance $2D_w$ apart. The normalized force of attraction is plotted in figure 5 against D_w/L. The ship geometry is the same as in example 2. The extension of the graph towards small values of D_w/L. is limited by the assumption stated at the outset of § 5 that $D = O(L)$ in order that each vessel be in the 'outer field' of the other. The relative errors thereby introduced are of order h/D which was regarded as $O(\epsilon)$ but could have been defined as a second small parameter, particularly with a view to considering smaller values of D.

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